

# Corrigendum: Recover the source and initial value simultaneously in a parabolic equation

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In the paper above, Theorem 2 and its proof are incorrect. Because the function  $l(t)$  in (2.3) never vanishes at  $t = T$ , the key Lemma 1 is not cited correctly, so we cite Lemma 2.4 in [3] as our key lemma in this corrigendum. Due to this consideration, we need to widen the bounded domain  $Q = \Omega \times (0, T)$  in (1.1) to  $Q = \Omega \times (0, T + \delta_0)$ , where  $\delta_0$  is an arbitrary fixed positive constant, i.e., we consider the following parabolic problem

$$\begin{cases} u_t = Au + f(x, t), & \text{in } Q = \Omega \times (0, T + \delta_0), \\ \frac{\partial u}{\partial \nu_A} = 0, & \text{on } \partial\Omega \times (0, T + \delta_0), \\ u(x, 0) = g(x), & \text{in } \Omega \end{cases} \quad (0.1)$$

in Theorem 2, where  $A$  is a uniformly elliptic operator of second order with  $x$ -dependent coefficients, and  $\frac{\partial u}{\partial \nu_A}$  is the conormal derivative with respect to  $u$ . The admissible set is given by

$$U = \left\{ (f, g) \mid (f, g) \in C^{2+\gamma, \frac{2+\gamma}{2}}(\overline{Q}) \times C^{4+\gamma}(\overline{\Omega}); \|f\|_{C^{2+\gamma, \frac{2+\gamma}{2}}(\overline{Q})} + \|g\|_{C^{4+\gamma}(\overline{\Omega})} \leq M_0 \right\},$$

( $0 < \gamma < 1$ ),

and the source function  $f$  in (0.1) satisfies

$$|f_t(x, t)| \leq C_0 |f(x, T)|, \quad (x, t) \in \overline{Q}, \quad (0.2)$$

for some positive constant  $C_0$ . Then the modified function  $l(t) = t(T + \delta_0 - t)$

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in (0.3) vanishes at  $t = T + \delta_0$ . Moreover,

$$\rho(x, t) = \frac{e^{\lambda\psi(x)}}{l(t)}, \quad \theta(x, t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi(x)\|_{C(\Omega)}}}{l(t)}, \quad (0.3)$$

and  $\psi(x)$  is defined in [3] Lemma 2.3. As for the detail proof of existence of  $\psi(x)$ , one can refer to Lemma 2.3 in [3]. Especially, the boundary measurement is modified to  $u|_{\Gamma \times (T-\delta_1, T+\delta_1)}$  ( $\delta_1$  will be defined later). Compared with the common measurement  $u|_{\Gamma \times (0, T+\delta_0)}$ , the measurement time is a subset of whole time interval  $(0, T + \delta_0)$ , which is more widely used in many applications. By doing above modification and following [3], we can actually obtain the Lipschitz stability for the source.

All these modifications are only used in Section 2 for obtaining the conditional stability and uniqueness, i.e. Theorem 2, Theorem 4 and Corollary 5. Because Theorem 4 and Corollary 5 are direct results of Theorem 2, we just focus on the corrections for Theorem 2, then the corrections for Theorem 4 and Corollary 5 are similar. In the other sections, the parabolic problem (0.1) is still considered in bounded domain  $Q = \Omega \times (0, T)$  and boundary measurement is  $u|_{\Gamma \times (0, T)}$ .

**Lemma 1** [3] *There exists a number  $\hat{\lambda} > 0$  such that for an arbitrary  $\lambda \geq \hat{\lambda}$  we can choose  $s_0(\lambda)$  such that for all  $s \geq s_0(\lambda)$ , the solution of parabolic problem (0.1)  $u(f, g) \in W_2^{2,1}(Q)$  satisfies the following inequality*

$$\begin{aligned} \int_Q \left( (s\rho)^{p-1} \left( |\partial_t u(f, g)|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u(f, g)|^2 \right) + (s\rho)^{p+1} |\nabla u(f, g)|^2 \right. \\ \left. + (s\rho)^{p+3} |u(f, g)|^2 \right) e^{2s\theta} dx dt \leq C \int_Q (s\rho)^p |f|^2 e^{2s\theta} dx dt \\ + C \int_{\Gamma \times (0, T+\delta_0)} \left( (s\rho)^p |\partial_t u(f, g)|^2 + (s\rho)^{p+1} |\nabla u(f, g)|^2 \right. \\ \left. + (s\rho)^{p+3} |u(f, g)|^2 \right) dS dt \quad p = 0, 1 \quad (0.4) \end{aligned}$$

where the constant  $C$  depends on  $\lambda$ , but independent of the large parameter  $s$ .

**Theorem 2.** (Conditional Stability) For every  $\delta_1 \in (0, \min\{\delta_0, T\}]$  and  $(f, g) \in U$ , let  $u(f, g)$  be the solution of (0.1), then we have

$$\begin{aligned} (1) \quad & \|f\|_{L^2(Q)} \leq C \left\| (u(f, g)(\cdot, T), u(f, g)) \right\|_{H^2(\Omega) \times H^2(\Gamma \times (T-\delta_1, T+\delta_1))}; \\ (2) \quad & \|g\|_{L^2(\Omega)} \leq C \left| \ln \left\| (u(f, g)(\cdot, T), u(f, g)) \right\|_{H^2(\Omega) \times H^2(\Gamma \times (T-\delta_1, T+\delta_1))} \right|^{-1}, \end{aligned}$$

where  $C$  is a positive constant, and

$$\begin{aligned} & \| (u(f, g)(\cdot, T), u(f, g)) \|_{H^2(\Omega) \times H^2(\Gamma \times (T - \delta_1, T + \delta_1))} \\ &= \left( \|u(f, g)(\cdot, T)\|_{H^2(\Omega)}^2 + \|u(f, g)\|_{H^2(\Gamma \times (T - \delta_1, T + \delta_1))}^2 \right)^{\frac{1}{2}} \end{aligned}$$

The proof of Theorem 2 is very similar to the one in [3], and we correct it as follows.

**PROOF.** (1) For every  $\delta_1 \in (0, \min\{\delta_0, T\}]$ , we have  $0 \leq T - \delta_1 < T < T + \delta_1 \leq T + \delta_0$ . Then we can construct weight functions as (0.3) in  $Q_1 = \Omega \times (T - \delta_1, T + \delta_1)$ , i.e.

$$\begin{aligned} l_1(t) &= (t - (T - \delta_1))((T + \delta_1) - t), \\ \rho_1(x, t) &= \frac{e^{\lambda\psi(x)}}{l_1(t)}, \\ \theta_1(x, t) &= \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi(x)\|_{C(\overline{\Omega})}}}{l_1(t)}, \end{aligned}$$

for  $(x, t) \in Q_1$ , where  $\psi(x)$  is defined in (0.3). Similar to [3], by the time transform  $\tilde{t} = t - T + \delta_1$ , we can change  $Q_1$  into  $\tilde{Q}_1 = \{(x, \tilde{t}) | (x, \tilde{t}) \in \Omega \times (0, 2\delta_1)\}$ , and change  $Q = \{(x, t) | (x, t) \in \Omega \times (0, T + \delta_0)\}$  into  $\tilde{Q} = \{(x, \tilde{t}) | (x, \tilde{t}) \in \Omega \times (-T + \delta_1, \delta_0 + \delta_1)\}$ . We focus on the domain  $\tilde{Q}_1$ , using the transform above, the weight functions become into

$$\begin{aligned} \tilde{l}_1(\tilde{t}) &= \tilde{t}(2\delta_1 - \tilde{t}), \\ \tilde{\rho}_1(x, \tilde{t}) &= \frac{e^{\lambda\psi(x)}}{\tilde{l}_1(\tilde{t})}, \\ \tilde{\theta}_1(x, \tilde{t}) &= \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi(x)\|_{C(\overline{\Omega})}}}{\tilde{l}_1(\tilde{t})}, \end{aligned}$$

for  $(x, \tilde{t}) \in \tilde{Q}_1$ . Setting  $\tilde{u}(x, \tilde{t}) := u(x, \tilde{t} + T - \delta_1)$  and  $\tilde{f}(x, \tilde{t}) := f(x, \tilde{t} + T - \delta_1)$ , then  $\tilde{u}$  in  $\tilde{Q}_1$  satisfies the following parabolic equation

$$\begin{cases} \tilde{u}_{\tilde{t}} = A\tilde{u} + \tilde{f}(x, \tilde{t}), & \text{in } \tilde{Q}_1, \\ \frac{\partial \tilde{u}}{\partial \nu_A} = 0, & \text{on } \partial\Omega \times (0, 2\delta_1), \\ \tilde{u}(x, 0) = u(x, T - \delta_1), & \text{in } \Omega. \end{cases} \quad (0.5)$$

Since  $(f, g) \in U$ , the solution of parabolic problem (0.1)  $u \in C^{4+\gamma, \frac{4+\gamma}{2}}(\overline{Q})$ . We

define  $v := \tilde{u}_{\tilde{t}}$ , then  $v_{\tilde{t}}$ ,  $Av$  exist and  $v$  satisfies

$$\begin{cases} v_{\tilde{t}} = Av + \tilde{f}_{\tilde{t}}(x, \tilde{t}), & \text{in } \tilde{Q}_1, \\ \frac{\partial v}{\partial \nu_A} = 0, & \text{on } \partial\Omega \times (0, 2\delta_1), \\ v(x, 0) = \tilde{u}_{\tilde{t}}(x, 0), & \text{in } \Omega. \end{cases} \quad (0.6)$$

Owing to  $\tilde{f}_{\tilde{t}} \in C^{\gamma, \frac{\gamma}{2}}(\overline{\tilde{Q}_1})$  and  $\tilde{u}_{\tilde{t}}(x, 0) \in C^{2+\gamma}(\overline{\Omega})$ , we see the solution of (0.6)  $v \in C^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\tilde{Q}_1})$ . Noting that the weight functions in  $\tilde{Q}_1$  are consistent with the ones in  $Q$ , we can still use the Carleman estimate (0.4) in  $\tilde{Q}_1$ . Thus, from Lemma 1 with  $p = 0$ , we get the Carleman inequality for  $v$ , that is, there exists  $\hat{\lambda} > 0$  such that for  $\lambda = \hat{\lambda}$  we can choose  $s_0(\hat{\lambda})$  such that for all  $s \geq s_0(\hat{\lambda})$ ,  $v$  satisfies

$$\begin{aligned} & \int_{\tilde{Q}_1} \left( \frac{1}{s\tilde{\rho}_1} \left( |\partial_{\tilde{t}}v|^2 + \sum_{i,j=1}^n |\partial_i\partial_jv|^2 \right) + s\tilde{\rho}_1|\nabla v|^2 + s^3\tilde{\rho}_1^3|v|^2 \right) e^{2s\tilde{\theta}_1} dx d\tilde{t} \\ & \leq C \int_{\tilde{Q}_1} |\tilde{f}_{\tilde{t}}(x, \tilde{t})|^2 e^{2s\tilde{\theta}_1} dx d\tilde{t} + C \int_{\Gamma \times (0, 2\delta_1)} \left( |\partial_{\tilde{t}}v|^2 + s\tilde{\rho}_1|\nabla v|^2 + s^3\tilde{\rho}_1^3|v|^2 \right) dS d\tilde{t}, \end{aligned} \quad (0.7)$$

where we set  $\lambda = \hat{\lambda}$  in  $\tilde{\rho}_1$ ,  $\tilde{\theta}_1$ , and throughout this section,  $C$  always denotes a positive generic constant which depends on  $\hat{\lambda}$ , but independent of large parameter  $s$ .

In particular, from above time linear transform, we find the measured time  $t = T$  is changed into  $\tilde{t} = \delta_1$ . Therefore, in  $\tilde{Q}$ , the condition (0.2) becomes into

$$|\tilde{f}_{\tilde{t}}(x, \tilde{t})| \leq C_0 |\tilde{f}(x, \delta_1)|, \quad (x, \tilde{t}) \in \overline{\tilde{Q}}. \quad (0.8)$$

Since  $\tilde{\theta}_1(x, \tilde{t}) \leq \tilde{\theta}_1(x, \delta_1)$ , for  $(x, \tilde{t}) \in \tilde{Q}_1$ , and from the condition (0.8), then (0.7) yields

$$\begin{aligned} & \int_{\tilde{Q}_1} \left( \frac{1}{s\tilde{\rho}_1} \left( |\partial_{\tilde{t}}v|^2 + \sum_{i,j=1}^n |\partial_i\partial_jv|^2 \right) + s\tilde{\rho}_1|\nabla v|^2 + s^3\tilde{\rho}_1^3|v|^2 \right) e^{2s\tilde{\theta}_1} dx d\tilde{t} \\ & \leq C \int_{\Omega} |\tilde{f}(x, \delta_1)|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx + C \int_{\Gamma \times (0, 2\delta_1)} \left( |\partial_{\tilde{t}}v|^2 + s\tilde{\rho}_1|\nabla v|^2 + s^3\tilde{\rho}_1^3|v|^2 \right) dS d\tilde{t}, \\ & \quad \forall s > s_0(\hat{\lambda}). \end{aligned} \quad (0.9)$$

According to  $v(x, \delta_1) = A_{\delta_1}\tilde{u} + \tilde{f}(x, \delta_1)$ , where

$$A_{\delta_1}q = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial q}{\partial x_j}(x, \delta_1) \right) + \sum_{i=1}^n b_i(x) \frac{\partial q}{\partial x_i}(x, \delta_1) + c(x)q(x, \delta_1), \quad (0.10)$$

we have

$$\begin{aligned} \int_{\Omega} s \left| \tilde{f}(x, \delta_1) \right|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx &\leq 2 \int_{\Omega} s \left| v(x, \delta_1) \right|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx \\ &\quad + 2 \int_{\Omega} s \left| A_{\delta_1} \tilde{u} \right|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx. \end{aligned} \quad (0.11)$$

Thanks to the construction of  $\tilde{\theta}_1(x, \tilde{t})$ , the following inequality is hold

$$\tilde{\theta}_1(x, \tilde{t}) \leq - \frac{\left( e^{2\hat{\lambda}\|\psi(x)\|_{C(\overline{\Omega})}} - e^{\hat{\lambda}\|\psi(x)\|_{C(\overline{\Omega})}} \right)}{\delta_1^2}, \quad (x, \tilde{t}) \in \overline{\tilde{Q}_1}. \quad (0.12)$$

writing  $M := \frac{\left( e^{2\hat{\lambda}\|\psi(x)\|_{C(\overline{\Omega})}} - e^{\hat{\lambda}\|\psi(x)\|_{C(\overline{\Omega})}} \right)}{\delta_1^2} > 0$ , by utilizing  $v^2(x, \tilde{t}) e^{2s\tilde{\theta}_1(x, \tilde{t})} \rightarrow 0$  ( $\tilde{t} \rightarrow 0^+$ ) and  $|\partial_{\tilde{t}} \tilde{\theta}_1| \leq C\tilde{\rho}_1^2$  in  $\overline{\tilde{Q}_1}$ , (0.11) implies

$$\begin{aligned} \int_{\Omega} s \left| \tilde{f}(x, \delta_1) \right|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx &\leq 2 \int_0^{\delta_1} \frac{\partial}{\partial \tilde{t}} \left( \int_{\Omega} s v^2(x, \tilde{t}) e^{2s\tilde{\theta}_1(x, \tilde{t})} dx \right) d\tilde{t} + C s e^{-2sM} \left\| \tilde{u}(x, \delta_1) \right\|_{H^2(\Omega)}^2 \\ &= 2 \int_{\Omega \times (0, \delta_1)} \left( 2s v(\partial_{\tilde{t}} v) + 2s^2 (\partial_{\tilde{t}} \tilde{\theta}_1) v^2 \right) e^{2s\tilde{\theta}_1(x, \tilde{t})} dx d\tilde{t} + C s e^{-2sM} \left\| \tilde{u}(x, \delta_1) \right\|_{H^2(\Omega)}^2 \\ &\leq 2 \int_{\Omega \times (0, \delta_1)} \left( \frac{2}{\sqrt{s\tilde{\rho}_1}} (\partial_{\tilde{t}} v) e^{s\tilde{\theta}_1} \right) \left( s \sqrt{s\tilde{\rho}_1} v e^{s\tilde{\theta}_1} \right) dx d\tilde{t} + C \int_{\Omega \times (0, \delta_1)} s^2 \tilde{\rho}_1^2 v^2 e^{2s\tilde{\theta}_1} dx d\tilde{t} \\ &\quad + C s e^{-2sM} \left\| \tilde{u}(x, \delta_1) \right\|_{H^2(\Omega)}^2 \\ &\leq 2 \int_{\Omega \times (0, \delta_1)} \frac{1}{s\tilde{\rho}_1} |\partial_{\tilde{t}} v|^2 e^{2s\tilde{\theta}_1} dx d\tilde{t} + 2 \int_{\Omega \times (0, \delta_1)} s^3 \tilde{\rho}_1 v^2 e^{2s\tilde{\theta}_1} dx d\tilde{t} \\ &\quad + C \int_{\Omega \times (0, \delta_1)} s^2 \tilde{\rho}_1^2 v^2 e^{2s\tilde{\theta}_1} dx d\tilde{t} + C s e^{-2sM} \left\| \tilde{u}(x, \delta_1) \right\|_{H^2(\Omega)}^2. \end{aligned}$$

Compare the last inequality with (0.9), it follows that

$$\begin{aligned} (s - C) \int_{\Omega} \left| \tilde{f}(x, \delta_1) \right|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx &\leq C \int_{\Gamma \times (0, 2\delta_1)} \left( |\partial_{\tilde{t}} v|^2 + s\tilde{\rho}_1 |\nabla v|^2 + s^3 \tilde{\rho}_1^3 |v|^2 \right) dS d\tilde{t} \\ &\quad + C s e^{-2sM} \left\| \tilde{u}(x, \delta_1) \right\|_{H^2(\Omega)}^2, \quad \forall s > s_0(\hat{\lambda}). \end{aligned} \quad (0.13)$$

On the other hand, in term of (0.8) in  $\tilde{Q}$ , we find

$$\begin{aligned} \int_{\tilde{Q}} |\tilde{f}(x, \tilde{t})|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx d\tilde{t} &= \int_{\tilde{Q}} \left| - \int_{\tilde{t}}^{\delta_1} \tilde{f}_{\xi}(x, \xi) d\xi + \tilde{f}(x, \delta_1) \right|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx d\tilde{t} \\ &\leq \int_{\tilde{Q}} \left( \left| \int_{\tilde{t}}^{\delta_1} \tilde{f}_{\xi}(x, \xi) d\xi \right| + \left| \tilde{f}(x, \delta_1) \right| \right)^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx d\tilde{t} \\ &\leq C \int_{\Omega} \left| \tilde{f}(x, \delta_1) \right|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx. \end{aligned} \quad (0.14)$$

From (0.13) and (0.14), it follows that

$$(s - C) \int_{\tilde{Q}} |\tilde{f}(x, \tilde{t})|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx d\tilde{t} \leq C \int_{\Gamma \times (0, 2\delta_1)} (|\partial_{\tilde{t}} v|^2 + s\tilde{\rho}_1 |\nabla v|^2 + s^3 \tilde{\rho}_1^3 |v|^2) dS d\tilde{t} \\ + C s e^{-2sM} \|\tilde{u}(x, \delta_1)\|_{H^2(\Omega)}^2, \quad \forall s > s_0(\hat{\lambda}). \quad (0.15)$$

Furthermore, setting  $s_1(\hat{\lambda}) := \max\{s_0(\hat{\lambda}), 2C\}$ , and we obtain

$$\frac{1}{2} \int_{\tilde{Q}} |\tilde{f}(x, \tilde{t})|^2 e^{2s\tilde{\theta}_1(x, \delta_1)} dx d\tilde{t} \leq \frac{1}{2} \int_{\Gamma \times (0, 2\delta_1)} (|\partial_{\tilde{t}} v|^2 + s\tilde{\rho}_1 |\nabla v|^2 + s^3 \tilde{\rho}_1^3 |v|^2) dS d\tilde{t} \\ + C e^{-2sM} \|\tilde{u}(x, \delta_1)\|_{H^2(\Omega)}^2, \quad \forall s > s_1(\hat{\lambda}). \quad (0.16)$$

In view of the continuity of  $\tilde{\theta}_1(x, \delta_1)$ , we see there exist a positive constant  $c_1(\hat{\lambda})$  such that  $\tilde{\theta}_1(x, \delta_1) \geq -c_1(\hat{\lambda}), \forall x \in \overline{\Omega}$ , and so

$$\int_{\tilde{Q}} |\tilde{f}(x, \tilde{t})|^2 dx d\tilde{t} \leq C s^3 e^{2c_1(\hat{\lambda})s} \int_{\Gamma \times (0, 2\delta_1)} (|\partial_{\tilde{t}} v|^2 + |\nabla v|^2 + |v|^2) dS d\tilde{t} \\ + C e^{2(c_1(\hat{\lambda})-M)s} \|\tilde{u}(x, \delta_1)\|_{H^2(\Omega)}^2, \quad \forall s > s_1(\hat{\lambda}). \quad (0.17)$$

Next we fix  $s$  in the right-hand side of (0.17), it concludes

$$\left( \int_{\tilde{Q}} |\tilde{f}(x, \tilde{t})|^2 dx d\tilde{t} \right)^{\frac{1}{2}} \leq C \left\| (\tilde{u}(\cdot, \delta_1), \tilde{u}(\cdot, \cdot)) \right\|_{H^2(\Omega) \times H^2(\Gamma \times (0, 2\delta_1))}. \quad (0.18)$$

Hence, noting the time inverse transform, we convert back to the  $t$ -variable and obtain (1).

(2) We directly write  $\vartheta := u_t$  in (0.1) and have

$$\begin{cases} \vartheta_t = A\vartheta + f_t(x, t), & \text{in } Q, \\ \frac{\partial \vartheta}{\partial \nu_A} = 0, & \text{on } \partial\Omega \times (0, T + \delta_0), \\ \vartheta(x, T) = A_T u + f(x, T), & \text{in } \Omega, \end{cases} \quad (0.19)$$

where the operator  $A_T$  is defined as (0.10). We decompose (0.19) as follows,

$$\begin{cases} w_t = Aw + f_t(x, t), & \text{in } Q, \\ \frac{\partial w}{\partial \nu_A} = 0, & \text{on } \partial\Omega \times (0, T + \delta_0), \\ w(x, 0) = 0, & \text{in } \Omega, \end{cases} \quad (0.20)$$

and

$$\begin{cases} z_t = Az, & \text{in } Q, \\ \frac{\partial z}{\partial \nu_A} = 0, & \text{on } \partial\Omega \times (0, T + \delta_0), \\ z(x, T) = A_T u + f(x, T) - w(x, T), & \text{in } \Omega. \end{cases} \quad (0.21)$$

Clearly,  $\vartheta = w + z$ , and then  $\vartheta(x, 0) = z(x, 0)$ , for all  $x \in \Omega$ . Similar to the solution of (0.6), we find  $\vartheta \in C^{2+\gamma, \frac{2+\gamma}{2}}(\overline{Q})$ . Consequently,

$$\|z(\cdot, 0)\|_{L^\infty(\Omega)} = \|\vartheta(\cdot, 0)\|_{L^\infty(\Omega)} \leq C\|\vartheta\|_{C^{2+\gamma, \frac{2+\gamma}{2}}(\overline{Q})} \leq CM_0.$$

Applying the well-know result (For example [4]), we have

$$\|z(\cdot, t)\|_{L^2(\Omega)} \leq (CM_0)^{1-\frac{t}{T}} \cdot \|z(\cdot, T)\|_{L^2(\Omega)}^{\frac{t}{T}}, \quad t \in [0, T].$$

Furthermore, by the semigroup theory (See [7]), we get

$$w(\cdot, t) = w(t) = \int_0^t S(t-\tau) f_\tau(\tau) d\tau,$$

where  $S(t)$ ,  $t \geq 0$  is the  $C_0$ -semigroup generated by  $A$ , and

$$D(A) = \left\{ u \in L^2(\Omega) \mid Au \in L^2(\Omega), \frac{\partial u}{\partial \nu_A} \Big|_{\partial\Omega} = 0 \right\}.$$

By the property of  $C_0$ -semigroup and condition (0.2), it follows that

$$\begin{aligned} \|w(\cdot, t)\|_{L^2(\Omega)} &= \int_0^t \|S(t-\tau)\| \cdot \|f_\tau(\cdot, \tau)\|_{L^2(\Omega)} d\tau \\ &\leq \int_0^t C \|f(\cdot, T)\|_{L^2(\Omega)} d\tau \\ &\leq C \|f(\cdot, T)\|_{L^2(\Omega)}, \end{aligned}$$

for all  $t \in [0, T]$ . Employing (0.21), we can estimate

$$\begin{aligned} \|\vartheta(\cdot, t)\|_{L^2(\Omega)} &\leq \|z(\cdot, t)\|_{L^2(\Omega)} + \|w(\cdot, t)\|_{L^2(\Omega)} \\ &\leq C \|z(\cdot, T)\|_{L^2(\Omega)}^{\frac{t}{T}} + C \|f(\cdot, T)\|_{L^2(\Omega)} \\ &\leq C (\|u(\cdot, T)\|_{H^2(\Omega)} + \|f(\cdot, T)\|_{L^2(\Omega)})^{\frac{t}{T}} + C \|f(\cdot, T)\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, utilize the  $L^2$ -estimation of  $u(\cdot, T)$  in (1) (such as (0.13)), and note that  $\|u(\cdot, T)\|_{H^2(\Omega)}$  and  $\|u\|_{H^2(\Gamma \times (T-\delta_1, T+\delta_1))}$  will be small enough, it implies

$$\begin{aligned} \|g\|_{L^2(\Omega)} &= \|u(\cdot, 0)\|_{L^2(\Omega)} \\ &= \left\| - \int_0^T \vartheta(\cdot, \tau) d\tau + u(\cdot, T) \right\|_{L^2(\Omega)} \\ &\leq C \int_0^T (\|u(\cdot, T)\|_{H^2(\Omega)} + \|f(\cdot, T)\|_{L^2(\Omega)})^{\frac{\tau}{T}} d\tau \\ &\quad + C \|f(\cdot, T)\|_{L^2(\Omega)} + \|u(\cdot, T)\|_{L^2(\Omega)} \\ &\leq C \frac{\left| 1 - \left\| (u(f, g)(\cdot, T), u(f, g)) \right\|_{H^2(\Omega) \times H^2(\Gamma \times (T-\delta_1, T+\delta_1))} \right|}{\left| \ln \left( \left\| (u(f, g)(\cdot, T), u(f, g)) \right\|_{H^2(\Omega) \times H^2(\Gamma \times (T-\delta_1, T+\delta_1))} \right) \right|} \\ &\quad + C \left\| (u(f, g)(\cdot, T), u(f, g)) \right\|_{H^2(\Omega) \times H^2(\Gamma \times (T-\delta_1, T+\delta_1))} \end{aligned}$$

$$\leq C \left| \ln \left( \left\| \left( u(f, g)(\cdot, T), u(f, g) \right) \right\|_{H^2(\Omega) \times H^2(\Gamma \times (T-\delta_1, T+\delta_1))} \right) \right|^{-1}.$$

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